

LOCALIZED STRUCTURES IN NONEQUILIBRIUM SYSTEMS

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We study numerically a prototype equation which arises generically as an envelope equation for a weakly inverted bifurcation associated to traveling waves: The complex quintic Ginzburg–Landau equation. We show six different stable localized structures including stationary pulses, moving pulses, stationary holes and moving holes, starting from localized initial conditions with periodic and Neumann boundary conditions.

Keywords: Oscillatory instability; Ginzburg–Landau equation; localized solutions.

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1. Introduction

In the last decade experimental evidence of localized structures in dissipative systems far from equilibrium has been reported. In a quasi one-dimensional system, an annulus filled with a mixture of ethanol and water and heated from below, localized structures of convection surrounded by non-convecting fluid has been studied.¹ More recently, the formation of clusters of localized structures via the self-completion scenario in a quasi two-dimensional gas discharge system,² and the interaction of dissipative localized structures in an optical pattern-forming system have been observed.³ Experiments on vertically vibrated granular layers in evacuated containers reveal a variety of patterns including particle-like localized excitations (oscillons).⁴ In chemical systems, catalytic oxidation of CO on Pt(110) exhibits oscillatory kinetics giving rise to solitary waves,⁵ and experiments on a ferrocyanide–iodate–sulfite reaction diffusion system show spot patterns that undergo a continuous process of growth through replication and death through overcrowding.⁶

The wide range of qualitatively different localized structures cannot be understood with a single mechanism. Coexistence between two stable states (not necessarily homogeneous⁷) or excitability are common features of the dynamics of nonequilibrium media that facilitates the formation of localized patterns.

Reaction-diffusion models have been successfully showing a rich variety of behaviors, such as self-replication,^{8,9} elastic behavior upon collision,^{10,11} or soliton behavior.¹² Localized solutions have been observed in monostable and bistable systems with two stable fixed points and one unstable fixed point¹³ or one stable fixed point, a stable limit cycle and an unstable limit cycle.^{12,14,15}

Localized solutions, like pulses, and their interactions have also been studied within the framework of envelope equations,^{16–20} and order parameters equations.^{21,22} In the domain of the envelope equations, the quintic complex Ginzburg-Landau equation is known to admit stable localized solutions like pulses as a consequence of the coexistence between a stable limit cycle and a stable fixed point and its non-variational nature.

The aim of this article is, on the one hand, to study numerically this equation elucidating the selection problem: Which solution will be reached starting from specified localized initial conditions and specified boundary conditions? And on the other hand, to show to what extent this equation with only one non-variational parameter (all parameters real except by one) is able to accept different classes of localized solutions.

2. Localized Solutions in the One-Dimensional Quintic Complex Ginzburg–Landau Equation (QCGLE)

The QCGLE represents an important prototype equation, since it arises generically as an envelope equation for a weakly inverted bifurcation associated to traveling waves. The one-dimensional QCGLE including dissipation and dispersion can be written as

$$\partial_t A = \mu A + \beta |A|^2 A + \gamma |A|^4 A + D \partial_{xx} A. \tag{1}$$

The subscript x denotes partial derivative with respect to x , $A(x, t) = r(x, t)e^{i\varphi(x, t)}$ is a complex field, and the parameters $\beta = \beta_r + i\beta_i$, $\gamma = \gamma_r + i\gamma_i$, and $D = D_r + iD_i$ are in general complex. The signs of the parameters $\beta_r > 0$ and $\gamma_r < 0$ are chosen in order to guarantee that the bifurcation is subcritical and saturates to quintic order. The control parameter μ is considered real. Equation (1) admits a class of homogeneous time-periodic solutions

$$A_{1,2} = r_{1,2} e^{i([\beta_i r_{1,2}^2 + \gamma_i r_{1,2}^4]t + \varphi_0)}, \tag{2}$$

where $r_{1,2}^2 = \beta_r \pm \sqrt{\beta_r^2 + 4|\gamma_r|\mu/2|\gamma_r|}$ and φ_0 is an arbitrary phase. The existence of $A_{1,2}$ requires that $\mu \geq -\beta_r^2/4|\gamma_r|$. However inside this range only A_1 is stable against small perturbations. It is easy to see that $A_0 = 0$ is also a solution of Eq. (1) but it is stable only for $\mu < 0$. Therefore the stable solutions A_0 and A_1 coexist

for $-\beta_r^2/4|\gamma_r| \leq \mu \leq 0$. Inside this coexistence range is where we are looking for localized solutions.

It is well known that if $\beta_i = \gamma_i = D_i = 0$ then Eq. (1) can be written as

$$\partial_t A = -\frac{1}{2} \frac{\delta \Phi}{\delta A^*}, \quad (3)$$

where A^* denotes the complex conjugate of A and $\Phi(\{A, A^*\})$ is a real functional

$$\Phi(\{A, A^*\}) = -2 \int \left(\mu |A|^2 + \frac{\beta_r}{2} |A|^4 + \frac{\gamma_r}{3} |A|^6 - D_r |\partial_x A|^2 \right) dx. \quad (4)$$

One can show that $\Phi(\{A, A^*\})$ is a Lyapounov function for Eq. (1) since multiplying Eq. (3) by $\delta \Phi / \delta A$, adding the complex conjugate and integrating we obtain the Lyapounov property

$$\frac{d\Phi}{dt} = \int \left(\partial_t A \frac{\delta \Phi}{\delta A} + c.c. \right) dx = - \int \left| \frac{\delta \Phi}{\delta A} \right|^2 dx \leq 0. \quad (5)$$

Due to the existence of a Lyapounov functional (free energy), when $\beta_i = \gamma_i = D_i = 0$, we call Eq. (1) *variational* quintic real Ginzburg–Landau equation. This consequently leads to an absence of a remaining dynamics in the system once an attractor is reached. The static case has been studied for critical phenomena near the superfluid phase transition of Helium II and for tricritical points.²³

In the general case ($b_i \neq 0$, $\gamma_i \neq 0$, $D_i \neq 0$), which arises generically in systems outside of equilibrium in the vicinity of an inverted Hopf bifurcation, there is no “free energy” to minimize. In this case we call Eq. (1) *non-variational* quintic complex Ginzburg–Landau equation.

Thus, in general the phase diagram of solutions of Eq. (1) involves seven parameters. Scaling the amplitude A , the time t and the space x we can fix three parameters and the problem is reduced to a 4-dimensional parameter space. Due to the complexity of the problem we set γ and D real and $\beta = \beta_r + i\beta_i$. Therefore the equation remains non-variational (due to the existence of β_i) and we are looking for localized solutions in a 2-dimensional parameter space (μ, β_i) . Throughout the rest of the article we use the parameter values $\beta_r = 1.125$, $\gamma_r = -0.859375$ and $D_r = 1$.

2.1. Periodic boundary conditions

We carry out a numerical analysis of Eq. (1) with periodic boundary conditions using fourth order Runge–Kutta finite differencing, a box length $L = 240$ with $dx = 0.4$, and a time step $dt = 0.1$. We are performing up to 10^6 iterations to check for long transients. None of the results depends sensitively on the discretization used.

We use two classes of initial conditions: ICP (initial conditions in phase) and ICA (initial conditions in antiphase) (see Fig. 1). The former is obtained by using $\text{Im } A(x) = 0$ and localized $\text{Re } A(x)$ positive (or negative) and the latter by choosing $\text{Im } A(x) = 0$ and $\text{Re } A(x)$ with a positive and a negative part. We note that none

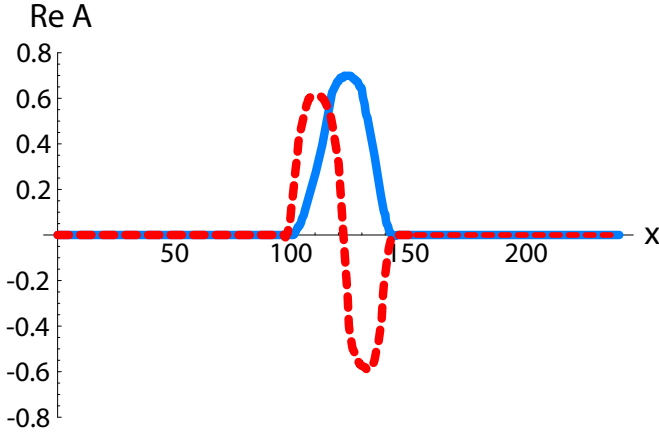


Fig. 1. Generic localized initial conditions: In phase (ICP) shown as solid line and in antiphase (ICA) shown as dashed line.

of the results presented below depends on the details of the shape of the initial conditions.

We are looking for localized solutions of Eq. (1) inside the range where $A = 0$ and the homogeneous solution coexist. Thus the localized initial conditions must be big enough (like Fig. 1) in order to overcome the basin of attraction of $A = 0$.

Fixing μ we proceed to look for stable localized solutions as a function of the non-variational parameter β_i and the class of initial conditions (ICP and ICA). For $\mu = -0.06$ and for ICP, the results are: In the range $0 < \beta_i < 0.456$, the system evolves to a homogeneous solution. For $0.456 < \beta_i < 0.487$, any ICP gives rise to a stationary 2π -hole (see Fig. 2(a)). We call this localized structure a 2π -hole because the modulus at the deepest part is not touching zero so that there is no jump in the phase. In the very wide range $0.487 < \beta_i < 1.9$, we obtain stationary pulses (see Fig. 2(b)). For $\beta_i > 1.9$ the system goes to zero.

For $\mu = -0.06$ and for ICA the results are: In the range $0 < \beta_i < 0.307$, the system evolves to a homogeneous solution. For $0.307 < \beta_i < 0.486$ any ICA generates an instance of the first type of stationary π -hole (see Fig. 2(c)). We call this localized solution a π -hole because the modulus reaches zero leading to a phase jump $\Delta\varphi = \pi$ around the cusp. Stationary 2π -holes appear in the narrow range $0.486 < \beta_i < 0.495$. In the very narrow range $0.495 < \beta_i < 0.496$ the system evolves to a second type of stationary π -holes (see Fig. 2(d)). Both the first and second type of stationary π -holes have qualitatively different slopes around $|A| = 0$. For $0.496 < \beta_i < 0.501$, we obtain a homogeneous solution. For $0.501 < \beta_i < 0.503$, right- or left-moving holes are generated, which are asymmetric π -holes and whose modulus reaches zero (see Fig. 2(e)). In the range $0.503 < \beta_i < 0.507$, any ICA evolves either to a right- or a left-moving pulse, which are asymmetric and have fixed shape (see Fig. 2(f)). In the very wide range $0.507 < \beta_i < 1.9$, any ICA reaches a stationary pulse. For $\beta_i > 1.9$ the system goes to zero.

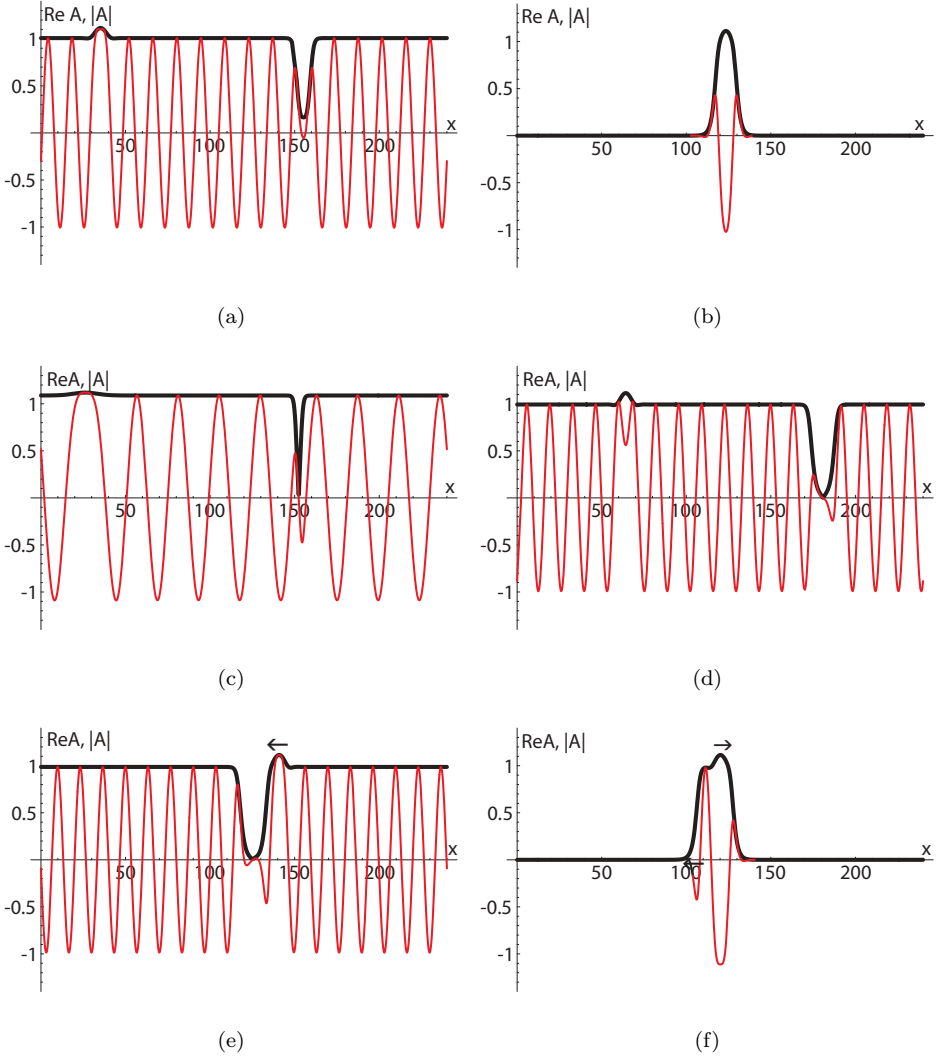


Fig. 2. Values of the parameters are: $\mu = -0.06$; $\beta_r = 1.125$; $\gamma_r = -0.859375$; $D_r = 1$. Thin continuous line represents $\text{Re } A$ and thick line stands by the modulus of the hole $|A|$. (a) Stationary 2π -hole for $\beta_i = 0.480$. (b) Stationary pulse for $\beta_i = 0.55$. (c) First type of π -hole for $\beta_i = 0.4$. (d) Second type of stationary π -hole for $\beta_i = 0.495$. (e) Left-moving π -hole for $\beta_i = 0.502$. (f) Right-moving pulse for $\beta_i = 0.505$.

For negative β_i we obtain qualitative similar results although the QCGLE is not invariant under the symmetry $\beta_i \rightarrow -\beta_i$.

2.2. From periodic to Neumann boundary conditions

In Fig. 2, we show the possible localized solutions in the QCGLE, with only one non-variational parameter, starting from localized initial conditions and periodic

boundary conditions (PBC). In this section, we address the question whether a change in the boundary conditions (from periodic to Neumann ($\partial_x A = 0$ at $x = 0, L$)) will lead to qualitative changes in the localized solutions created with PBC.

The results are the following: Starting from a stationary pulse as shown in Fig. 2(b) the change from PBC to Neumann boundary conditions (NBC) does not imply any modification in the pulse because asymptotically the pulse satisfies NBC. Now we start from a stationary 2π -hole as shown in Fig. 3(a) created by PBC. After changing from PBC to NBC, the hump in the modulus (source of traveling waves) starts moving to the boundaries in order to satisfy NBC, but the localized part of the hole (sink of traveling waves) remains at rest. At the end we obtain a new stationary 2π -hole (see Fig. 3(b)). The same behavior is observed when we start from π -holes of the first and second types, as shown in Figs. 2(c) and 2(d). Starting from a (left or right) moving π -hole (see Fig. 3(c)) and after changing from PBC to NBC the hole starts expanding and at the end we obtain two stationary half-pulses

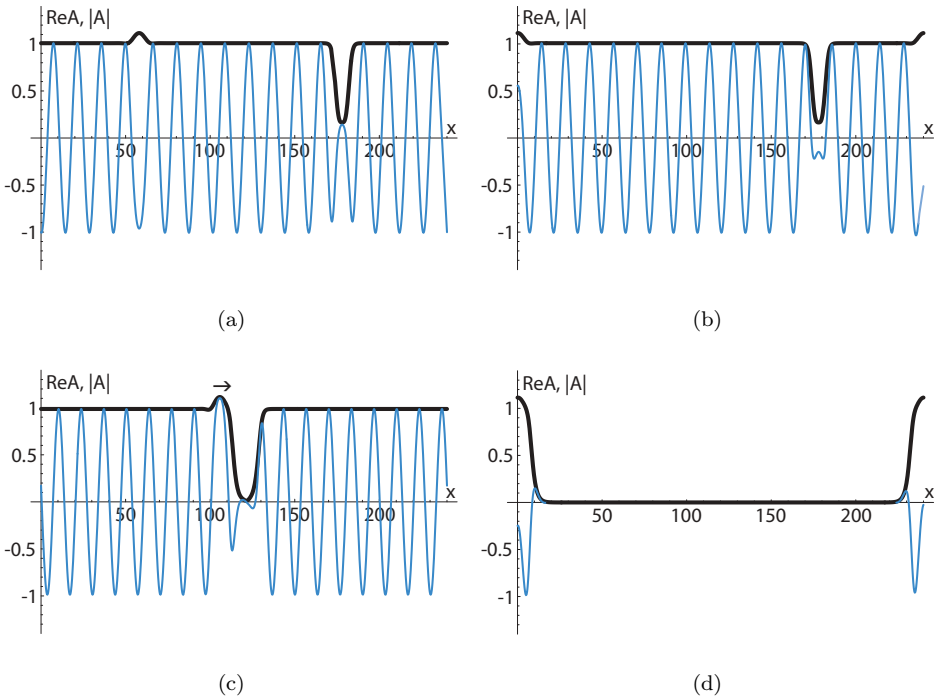


Fig. 3. Values of the parameters are: $\mu = -0.06$; $\beta_r = 1.125$; $\gamma_r = -0.859375$; $D_r = 1$. Thin continuous line represents $\text{Re } A$ and thick line stands by the modulus of the hole $|A|$. (a) Stationary 2π -hole for $\beta_i = 0.480$ created by PBC and localized initial condition. (b) Stationary 2π -hole for $\beta_i = 0.480$ after changing from PBC to NBC, using the initial condition of the 2π -hole shown in (a). (c) Right-moving π -hole for $\beta_i = 0.502$ created by PBC and localized initial condition. (d) Two stationary half-pulses after changing from PBC to NBC, using the initial condition of the right-moving π -hole shown in (c). (e) Stationary half-pulse after changing from PBC to NBC, using the initial condition of the left-moving pulse for $\beta_i = 0.505$. (f) Stationary half-pulse after changing from PBC to NBC using as initial condition, a right-moving pulse for $\beta_i = 0.505$.

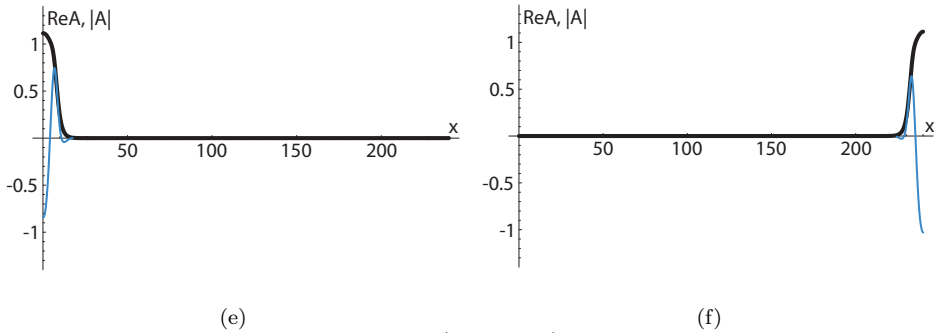


Fig. 3 (Continued).

(see Fig. 3(d)). A (left or right) moving pulse evolves to a stationary half-pulse (see Figs. 3(e) and 3(f)).

3. Conclusions

We studied the quintic complex Ginzburg-Landau equation numerically, with only one non-variational parameter. We have shown that this prototype equation, with periodic boundary conditions, admits six different stable localized structures: Stationary pulses, moving pulses, three kinds of stationary holes, and moving holes. So far, the existence (and even coexistence) of stationary pulses, stationary 2π -holes and π -holes of the first kind was known only for a simple reaction-diffusion model.²⁴ We obtained two different sequences of localized solutions starting from the different classes of localized initial conditions, namely, initial conditions in phase and initial conditions in antiphase. Finally we addressed the question whether a change in the boundary conditions (from periodic to Neumann) leads to qualitative changes in the localized solutions created with periodic boundary conditions.

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