Existence Range of Pulses in the Quintic Complex Ginzburg-Landau Equation

Pablo Gutiérrez* and Orazio Descalzi†

*Departamento de Física, FCFM, Universidad de Chile, Santiago, Chile
†Facultad de Ingeniería, Universidad de los Andes, Santiago, Chile

Abstract. Given a system in the vicinity of an oscillatory subcritical bifurcation (Hopf) that presents localized structures, we model its dynamics with the Quintic Complex Ginzburg-Landau Equation. For pulse-type structures, we study the bifurcation to fronts via a quasi-analytical approach.

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INTRODUCTION

In the last decades the importance of studying pulse-type localized structures has been shown, because they are observed in different physical contexts: hydrodynamics, chemical reactions, population dynamics, plasma physics, granular media, etc. The presence of pulses has been discovered particularly in systems whose dynamics is oscillatory, like binary-fluids [1] and chemical surface reactions [2].

In 1988, Thual and Fauve started a line of research on pulses showing the existence of pulses by using as a model the Quintic Complex Ginzburg-Landau Equation (QCGLE) [3]. The QCGLE arises generically as an envelope equation for a weakly inverted bifurcation associated with traveling waves. This equation has a non-variational nature and coexistence of stable solutions.

From that pioneer work a wide field of investigation was opened, finding new solutions for QCGLE [4–8] and in the creation of analytical methods that allow us understand the properties of those solutions. From an analytical point of view we can remark the limits of QCGLE, as the variational limit [9] or the perturbative analysis of solitons in the non-linear Schrödinger equation (conservative limit) [10]. Recently Descalzi et al. proposed a quasi-analytical approach that approximate pulses on the whole range between the above referred limits. In addition, they show that the appearance of pulses is related to a saddle-node bifurcation [11, 12].

In this paper, we focus on the pulse-front bifurcation and, in this way, it is possible characterize the existence range of pulses.

QUASI-ANALYTICAL STUDY OF PULSES

Our starting point is the QCGLE:

$$\partial_t A = \mu A + \beta |A|^2 A + \gamma |A|^4 A + D\partial_{xx} A, \tag{1}$$

where the subscripts x and t denote partial derivatives with respect to space and time respectively. $A(x,t) = r(x,t)e^{i\phi(x,t)}$ is a complex field. The control parameter μ is considered as real without losing generality. The parameters $\beta = \beta_r + i\beta_i$, $\gamma = \gamma_r + i\gamma_i$ and $D = D_r + iD_i$ are in general complex and contain the physical information of the problem under study. In order to assure that the bifurcation is subcritical and saturates the quintic order it is necessary to demand $\beta_r > 0$ and $\gamma_r < 0$. For the homogeneous case, the system shows coexistence of stable solutions in the range $\beta_r^2/4\gamma_r \leq \mu \leq 0$.

If the localized structure is stationary which means r is a function depending only on the space, then it is possible to make the following Ansatz [3]:

$$A(x,t) = R_0(x)\exp\{i(\Omega t + \theta_0(x))\}, \tag{2}$$

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where Ω is a unknown parameter. Replacing the Ansatz in (1) one obtain two real equations, and after a simple algebra results:

$$0 = \mu_+ R_0 + \beta_+ R_0^3 + \gamma_+ R_0^5 + R_{0xx} - R_0 \theta_{0x}^2, \tag{3}$$

$$\mu_- R_0 = \beta_- R_0^3 + \gamma_- R_0^5 + 2R_{0x}\theta_{0x} + R_0 \theta_{0xx}, \tag{4}$$

with

$$\begin{aligned} \mu_+ &= \tfrac{D_r\mu - D_i\Omega}{|D|^2}, \quad \beta_+ = \tfrac{D_r\beta_r + D_i\beta_i}{|D|^2}, \quad \gamma_+ = \tfrac{D_r\gamma_r + D_i\gamma_i}{|D|^2}, \\ \mu_- &= \tfrac{D_i\mu + D_r\Omega}{|D|^2}, \quad \beta_- = \tfrac{D_r\beta_i - D_i\beta_r}{|D|^2}, \quad \gamma_- = \tfrac{D_r\gamma_i - D_i\gamma_r}{|D|^2}, \end{aligned} \tag{5}$$

and $|D|^2 = D_r^2 + D_i^2$. It is important to notice that if θ_{0x} is constant, one can integrate explicitly the equation (3). If one observe the associated numerical simulations [7, 8], becomes clear the validity of this condition on θ_{0x} in almost the whole space, but around the centre of the localized structure. This numerical fact allowed Descalzi et. al to propose the approximation scheme introduced in [11, 12]: the space is divided in one region near the center of the pulse (*core*) and another region out of the center of the pulse (*outside the core*).

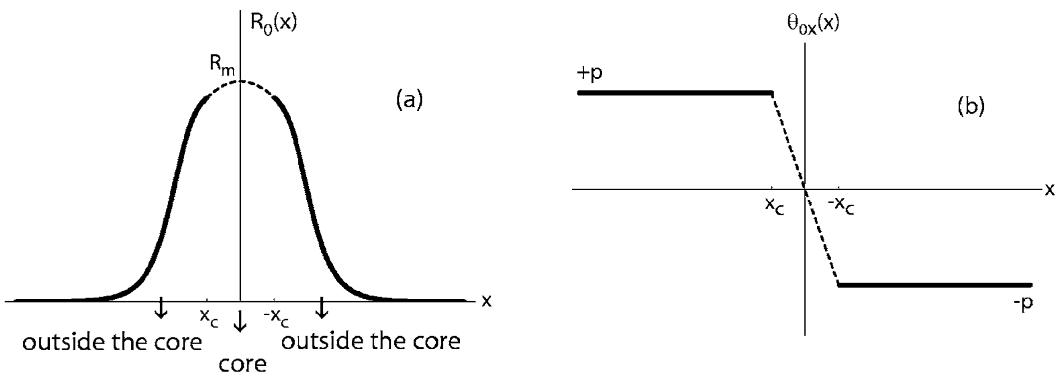

FIGURE 1. Regions of approximation for a pulse. The thick line is the *outside the core* region. The dotted line is the *core* region. (a) Modulus of the pulse R_0. (b) Phase gradient θ_{0x}.

Inside the *core* one approximate the functions $R_0(x)$ and $\theta_{0x}(x)$ by their first terms in Taylor expansion writing:

$$R_0(x) = R_m - \varepsilon x^2, \quad \theta_{0x} = -\alpha x, \tag{6}$$

where $(Rm, \varepsilon, \alpha)$ are unknown parameters. R_m is the maximum height of the pulse at $x = 0$.

When the equations (6) are included in (3) and (4) it is obtained: $\varepsilon = (\mu_+ R_m + \beta_+ R_m^3 + \gamma_+ R_m^5)/2$ and $\alpha = \beta_- R_m^2 + \gamma_- R_m^4 - \mu_-$. ε and α are now expressed in terms of (R_m, Ω) and of the original parameters of the equation (1).

Outside the core, where θ_{0x} is constant ($+p$ for $x < x_c$ and $-p$ for $x > x_c$), one can integrate explicitly equation (3). The solution is:

$$R_0(x) = \frac{2b^{\frac{1}{4}} \exp\{\sqrt{-\mu_+ + p^2}(|x| + x_0)\}}{\sqrt{(\exp\{2\sqrt{-\mu_+ + p^2}(|x| + x_0)\} + \frac{a}{\sqrt{b}})^2 - 4}}, \tag{7}$$

where $a = -3\beta_+/2\gamma_+$, $b = -3(-\mu_+ + p^2)/\gamma_+$ and x_0 is a constant that emerges from the translation symmetry of the QCGLE.

Ω can be asymptotically evaluated and we obtain: $\Omega = (-D_i(\mu D_r - 2p^2|D|^2) + 2p|D|^2\sqrt{-\mu D_r + p^2|D|^2})/D_r^2$, that only depends on p (and on the parameters of (1)). Now ε and α depends on (Rm, p).

Next step in the Matching Approach is to impose continuity to the functions at $x_c = -p/\alpha$. The continuity in $R_0(x)$ fixes the value of x_0 in terms of R_m and p:

$$x_0 = x_c + \frac{\ln u_*}{\sqrt{-\mu_+ + p^2}}, \tag{8}$$

where

$$u_*^2 = -\frac{a}{\sqrt{b}} + \frac{2\sqrt{b}}{r_c^2} + \frac{2}{r_c^2}\sqrt{r_c^4 - a r_c^2 + b} \tag{9}$$

and $r_c = R_m - \varepsilon x_c^2$.

The continuity of $R_{0x}(x)$ at $x = x_c$ gives us the first relation between R_m and p:

$$f(Rm,p) \equiv \sqrt{-\frac{\gamma_+}{3}} r_c \sqrt{r_c^4 - a r_c^2 + b} + 2\varepsilon x_c = 0. \tag{10}$$

A second relation emerges from a condition of consistency, introduced in [3], obtained multiplying equation (4) by $R_0(x)$ and integrating the real axis. Since $R_0(x)$ is a symmetric function, the relation is reduced to:

$$g(R_m,p) \equiv \mu_- \int_{-\infty}^{0} R_0^2 dx - \beta_- \int_{-\infty}^{0} R_0^4 dx - \gamma_- \int_{-\infty}^{0} R_0^6 dx = 0. \tag{11}$$

The integrals can be evaluated and the result is:

$$\int_{-\infty}^{0} R_0^2 dx = \frac{1}{2}\sqrt{-\frac{3}{\gamma_+}} \ln\left|\frac{a+\sqrt{b}(u_*^2+2)}{a+\sqrt{b}(u_*^2-2)}\right| - R_m^2 x_c + \frac{2}{3} R_m \varepsilon x_c^3. \tag{12}$$

$$\int_{-\infty}^{0} R_0^4 dx = -\sqrt{-\frac{3}{\gamma_+}} \frac{(a^2-4b)\sqrt{b} + abu_*^2}{(-4b+(a+\sqrt{b}u_*^2)^2)} + \frac{a}{4}\sqrt{-\frac{3}{\gamma_+}} \ln\left|\frac{a+\sqrt{b}(u_*^2+2)}{a+\sqrt{b}(u_*^2-2)}\right| - R_m^4 x_c + \frac{4}{3} R_m^3 \varepsilon x_c^3. \tag{13}$$

$$\begin{aligned}
\int_{-\infty}^{0} R_0^6 dx \;=\; & \frac{\sqrt{-\frac{3}{\gamma_+}}}{16(-4b+(a+\sqrt{b}u_*^2)^2)^2} \Big(-12a\sqrt{b}(a^2-4b)^2 - 4bu_*^2(a^2-4b)(9a^2+4b) - 12a(3a^2-4b)b^{\frac{3}{2}} u_*^4 \\
& +4b^2(-3a^2+4b)u_*^6 + (3a^2-4b)(-4b+(a+\sqrt{b}u_*^2)^2)^2 \ln\left|\frac{a+\sqrt{b}(u_*^2+2)}{a+\sqrt{b}(u_*^2-2)}\right| \Big) \\
& -R_m^6 x_c + 2R_m^5 \varepsilon x_c^3.
\end{aligned} \tag{14}$$

The construction of relations $f(R_m,p) = 0$ and $g(R_m,p) = 0$ is essential because the existence of the pulses is conditioned by them: the pulse exist if the matching takes place. This occurs when the relations are simultaneously fulfilled and graphically implies the intersection between the curves that were defined in the (R_m,p) plane.

Through this procedure the mechanism of pulses appearance has been explained: there is a critical value μ_{c1}, so that for$\mu < \mu_{c1}$ the curves $f = 0$ and $g = 0$ do not intersect at any point suggesting that there are no pulses. For $\mu > \mu_{c1}$ the curves intersect in two points, that represent stable and an unstable pulses. This means that the appearance mechanism of pulses is a saddle-node bifurcation [11, 12].

MECHANISM OF STABLE PULSES DISAPPEARANCE

If μ is further increased, then the numerical observations [7] show that the stable pulse disappears leading to the emergence of fronts [13]. Therefore, there is a second interesting critical μ (μ_{c2}).

From the point of view of the Matching Approach the disappearance of pulses is related to the restrictions imposed by x_0 (obtained in (8)), since it should be real. It is only in this way that the expression (7) of $R_0(x)$ has sense.

The restriction in x_0 results from u_* which is the result of the evaluation of $\sqrt{u_*^2}$. Hence, u_*^2 should be positive. The condition $u_*^2 = 0$ can be rewritten as $a^2 = 4b$ (near the intersection of $f = 0$ and $g = 0$). Then, p can be fixed in a constant value p_{crit} through direct algebra. Next, x_0 is real in the region $p > p_{crit}$ (near the intersection).

$$p_{crit} = \sqrt{\frac{D_r\mu - \alpha_1 + \sqrt{\alpha_2 - \alpha_3\mu}}{D_r^2}}, \tag{15}$$

with

$$\alpha_1 = \frac{3}{16}\frac{(D_r\beta_r + D_i\beta_i)^2}{D_r\gamma_r + D_i\gamma_i}\left(1 + \frac{D_i^2}{|D|^2}\right), \quad \alpha_2 = \left(\frac{3}{8}\frac{(D_r\beta_r + D_i\beta_i)^2}{D_r\gamma_r + D_i\gamma_i}\frac{D_i}{|D|}\right)^2, \quad \alpha_3 = \frac{3}{4}\frac{(D_r\beta_r + D_i\beta_i)^2}{D_r\gamma_r + D_i\gamma_i}\frac{D_rD_i^2}{|D|^2}. \tag{16}$$

Written in this way, it is clear that p_{crit} depends only on the parameters of (1), particularly of μ.

In terms of μ, μ_{c2} sets the existence of the pulse (Fig. 2). For $\mu < \mu_{c2}$ there is an intersection of $f = 0$ and $g = 0$ where $p > p_{crit}$. This means that there exists a stable pulse (Fig. (2a) and (2b)). For $\mu > \mu_{c2}$ the intersection of $f = 0$ and $g = 0$ is obtained in $p < p_{crit}$. In this case there is an absence of pulse (Fig. 2c). It is necessary to notice that μ_{c2} is such that the intersection occurs in $p = p_{crit}$.

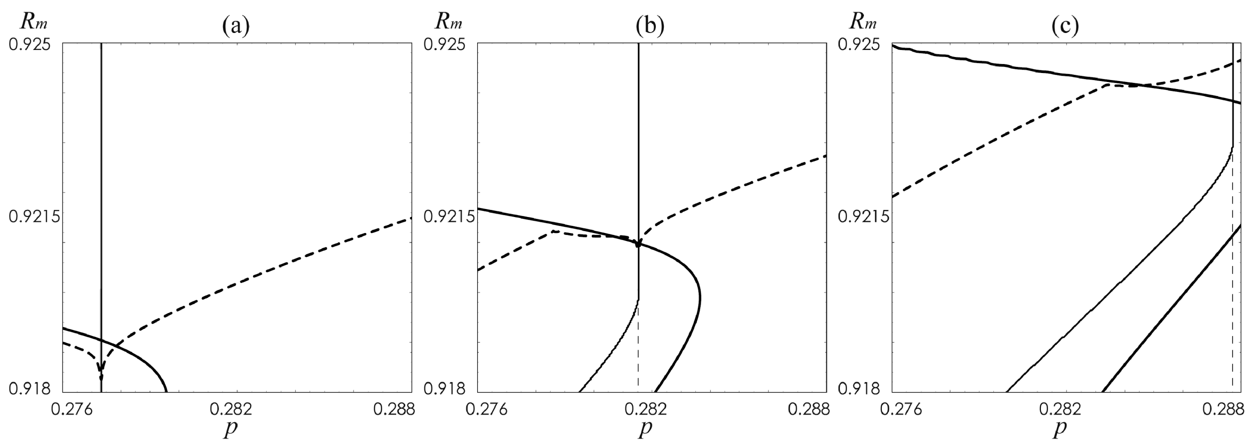

FIGURE 2. The (R_m, p) plane for different μ. The thick continuous line corresponds to $f = 0$. The thick dotted line corresponds to $g = 0$. The thin line determines the validity of x_0: in the right side x_0 is real. The thin dashed line corresponds to $p = p_{crit}$. If the thick constant and dotted lines are intersected with $p > p_{crit}$, then there is stable pulse. The parameters used in every figure are: $\beta_r = 1$, $\beta_i = 0.2$, $\gamma_r = -1$, $\gamma_i = 0.15$, $D_r = 1$, $D_i = -0.1$. (a) $\mu = -0.122 < \mu_{c2}$, the intersection of $f = 0$ and $g = 0$ is produced with $p = 0.277841 > p_{crit}(\mu) = 0.277313$. Therefore, there exists pulse. (b) $\mu = -0.120 < \mu_{c2}$, the intersection is produced with $p = 0.281575 > p_{crit}(\mu) = 0.281523$. There is pulse too. (c) $\mu = -0.117 > \mu_{c2}$, there is an absence of pulse because the intersection occurs at $p = 0.284637 < p_{crit}(\mu) = 0.287704$.

OBTENTION OF μ_{c2}

For the Matching Approach all the variables and functions are written in only two free parameters R_m and p. however, the study of bifurcations has always included μ playing the same role. This is so because in the physical interpretation μ is a control parameter. For instance, in an Rayleigh-Bérnard type experiment μ is proportional to the difference of temperatures ΔT among the plates of the fluid layer.

From this viewpoint, the functions f and g can be written as $f(R_m, p, \mu)$ and $g(R_m, p, \mu)$. On the other hand, it was already mentioned that the existence of pulses is determined by the condition $p = p_{crit}(\mu)$. This condition eliminates one of the dependencies, and let us obtain μ_{c2} through the intersection of the functions, now in the (R_m, μ) plane.

When the definition (11) of g, the integrals (12), (13), (14) and the conditions fixing p_{crit} ($u_*^2 = 0$ and $a^2 = 4b$) are observed, it is clear that the function g is undefined in p_{crit}. Thus, and keeping in mind that the dominant terms of g are logarithms, it approaches to:

$$g_{aprox}(R_m,\mu) \equiv 8\mu_- - 4\beta_- a - \gamma_-(3a^2 - 4b) = 0, \tag{17}$$

which is a function only of μ. Consequently, μ_{c2} is obtained explicitly:

$$\mu_{c2} = \frac{\alpha_2 - (\alpha_1 - D_r\alpha_4)^2}{\alpha_3}, \tag{18}$$

with $\alpha_4 = \frac{3}{4}\frac{D_r\beta_r + D_i\beta_i}{D_r\gamma_r + D_i\gamma_i}\left(\beta_r - \frac{3}{4}\gamma_r\frac{D_r\beta_r + D_i\beta_i}{D_r\gamma_r + D_i\gamma_i}\right)$.

In order to fix ideas we use (in Figs. 2 and 3) parameters [13] from which wide numerical results are known [7]. These parameters are $\beta_r = 1$, $\beta_i = 0.2$, $\gamma_r = -1$, $\gamma_i = 0.15$, $D_r = 1$, $D_i = -0.1$. With them can be obtained: $\mu_{c1} = -0.16776$ for the appearance of pulses and $\mu_{c2} = -0.117066$ for their disappearance. When comparing the numerical simulations it is found that the method's error is approximately 1% and 5% respectively.

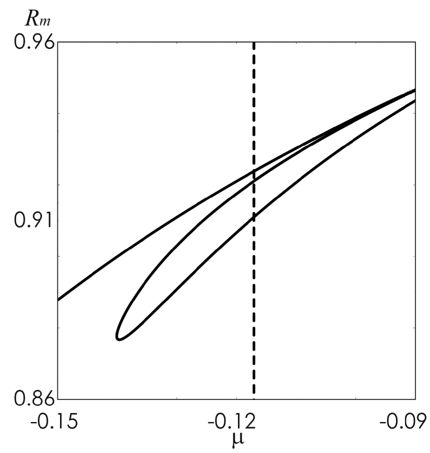

FIGURE 3. The (R_m,μ) plane. The thick continuous line corresponds to $f = 0$ and the thick dotted line corresponds to $g_{approx} = 0$, evaluated from (17). Using the same parameters than Fig. 2, $\mu_{c2} = -0.117066$ can be obtained.

CONCLUSIONS

In this article we have focused on pulses using the QCGLE as a model. This envelope equation arises generically near subcritical bifurcations for systems with oscillatory dynamics.

Firstly, we revised a quasi-analytical method to study pulses in a wide range of parameters. The method consists of approximating the shape of the pulse (its amplitude and phase gradient) separating the space into two regions: the core and outside the core, in order to impose continuity. With this scheme the appearance of pulses via saddle-node bifurcation was previously characterized, and one can obtain the minimum μ when the pulses exist.

Then, one looks for the maximum μ when the pulses exist. For that, when one increases μ, one must check the validity of the approximation that guarantee the pulse existence. In particular, one can see that when the approximated solution of the pulse is explicitly written, one of its variables (x_0) loses its physical sense. With this, we conclude that the pulse disappear leading to the emergence of fronts. The second μ of bifurcation (that is the maximum μ when the pulses exist) can be analytically approximated because the restrictions on x_0 simplify the matching conditions until μ_{c2} is fixed. When it is compared, μ_{c2} agree within 5% of recent numerical simulations.

As a perspective, it would be interesting to apply these results to the two dimensional QCGLE (trivial generalization), where a radial pulse solution was observed with a very different existence range although the same parameters are used.

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