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Collisions of counter-propagating pulses in coupled complex cubic-quintic Ginzburg–Landau equations

O. Descalzi[1,2,a], J. Cisternas[1], P. Gutiérrez[3], and H.R. Brand[2]

[1] Facultad de Ingeniería, Universidad de los Andes, Av. San Carlos de Apoquindo 2200, Santiago, Chile
[2] Department of Physics, University of Bayreuth, 95440 Bayreuth, Germany
[3] Departamento de Física, FCFM, Universidad de Chile, Casilla 487-3, Santiago, Chile

Abstract. We discuss the results of the interaction of counter-propagating pulses for two coupled complex cubic-quintic Ginzburg–Landau equations as they arise near the onset of a weakly inverted Hopf bifurcation. As a result of the interaction of the pulses we find in 1D for periodic boundary conditions (corresponding to an annular geometry) many different possible outcomes. These are summarized in two phase diagrams using the approach velocity, v, and the real part of the cubic cross-coupling, c_r, of the counter-propagating waves as variables while keeping all other parameters fixed. The novel phase diagram in the limit $v \to 0, c_r \to 0$ turns out to be particularly rich and includes bound pairs of 2π holes as well as zigzag bound pairs of pulses.

We dedicate this article to the memory of our friend and colleague Prof. C. Pérez-García.

1 Introduction

During the last decade careful experiments leading to collisions between localized structures have been carried out. For example, in a thin horizontal layer of a binary fluid mixture, which is heated from below, spatially restricted convecting regions coexist stably with non-convecting (conductive) regions. Kolodner [1] studied collisions between two counter-propagating pulses of convective traveling waves in an annular channel. As a function of the speed with which they approach one another he observed that at high speed only one pulse survives after the collision and at low speed a bound pair of pulses arises. In this connection it is worthwhile to keep in mind, that it was shown experimentally [2] as well as numerically [3] that localized pulses of traveling waves in binary-fluid mixtures are not weakly nonlinear structures because of the large-scale concentration flows. Therefore an envelope equation approach is not directly applicable. In this connection we also mention some recent work on localized states in binary fluid convection, which can be considered as a steady 'soliton' resulting from the interaction of traveling waves [4–6].

Another example is given by a surface reaction, the catalytic oxidation of carbon monoxide on a Pt(110) single-crystal surface. For the range of parameters, where oscillatory kinetics has been observed, Rotermund et al. [7] found that waves of enhanced oxygen coverage propagate with constant velocity. Collisions of counter-propagating pulses were mostly leading to

[a] e-mail: oraziod@uandes.cl

annihilation of only one pulse. The cases of mutual annihilation or interpenetration of two pulses occurred less frequently.

Modeling of various reaction–diffusion (RD) systems has shown a rich pulse dynamics. Using excitable RD systems, it was shown by several groups [8,9] that propagating pulses not necessarily annihilate but that they can preserve their shapes and velocity after a collision. Kosek and Marek [10] studied the interaction of stable pulses in a RD model of the Belousov–Zhabotinsky reaction valid for both, the cerium and ferroin catalyst. In a range of parameters close to a subcritical instability they found interpenetration of counter-propagating pulses. The above mentioned experiments [1,7] and the model studied by Kosek et al. [10] rely on an oscillatory subcriticality where two locally stable states coexist. Therefore one is tempted to study two coupled subcritical (cubic-quintic) complex Ginzburg-Landau (CGL) equations for counter-propagating waves, which arise as prototype envelope equations near the onset of a weakly inverted oscillatory instability against traveling waves. Indeed, Brand and Deissler [11] showed that complete interpenetration of two one-particle solutions can arise for a strongly dissipative system like coupled subcritical CGL equations. Moreover, in that work the authors reported annihilation of two one-particle solutions, interpenetration of a one- and a two-particle state, annihilation of a two-particle solution by a one-particle state and a partial annihilation of a two-particle state by a one-particle state. Later, the same authors [12] studied the interaction of two-dimensional solutions in the same system obtaining a stationary compound state as a result of a head-on collision for stabilizing cross coupling and sufficiently small group velocity. The annihilation of a one-particle solution by another one-particle has been shown only in the case where one of them has not reached its asymptotic shape before the collision [13].

CGL equations are generic in the sense that the equations are only related to the instability and the symmetries of the system under study. Nevertheless, some general assumptions are required to derive CGL equations, for instance, the validity of the weakly nonlinear approximation [14,15]. Thus, in this article we are considering coupled subcritical cubic-quintic CGL equations as a phenomenological dispersive-dissipative model supporting stable localized structures.

The paper is organized as follows. In section 2 we give a brief of what is known about localized solutions of a single cubic-quintic complex Ginzburg-Landau equation followed by a brief summary of our recent work [16] on coupled cubic-quintic complex Ginzburg-Landau equations. In section 4 we present for the first time the phase diagram in the limit of small group velocity and small positive cubic-cross coupling between counter-propagating waves along with a detailed description of zigzag bound pairs of pulses and counter-propagating 2π holes.

2 Single cubic-quintic complex Ginzburg–Landau equation

Before we focus on the coupled cubic-quintic CGL equations we present the state of current knowledge for a single one-dimensional cubic-quintic CGL equation, which reads

$$\partial_t A = \mu A + \beta |A|^2 A + \gamma |A|^4 A + D\partial_{xx} A. \tag{1}$$

The subscript x denotes partial derivative with respect to x, $A(x,t)$ is a complex field, and the parameters $\beta = \beta_r + i\beta_i$, $\gamma = \gamma_r + i\gamma_i$, and $D = D_r + iD_i$ are in general complex. The signs of the parameters $\beta_r > 0$ and $\gamma_r < 0$ are chosen in order to guarantee that the bifurcation is subcritical and saturates to quintic order. The control parameter μ is taken to be real without loss of generality.

Since Thual and Fauve [17] showed that a cubic-quintic CGL equation admits stable localized solutions, much work has been performed in studying pulse solutions in this equation.

From an analytical point of view we mention that in the conservative limit of this equation a perturbative analysis of solitons in the nonlinear Schrödinger equation has been carried out [18, 19]. The opposite limit (variational limit) has been studied in [20,21]. In particular in [20] the authors showed that stable pulses exist in a narrow range of parameters close to the Maxwell point. Later, Descalzi et al. [22], for a more general case, reported that the appearance mechanism of pulses in the cubic-quintic CGL equation is related to a saddle-node bifurcation. In addition, exact solitary wave solutions were obtained using a method derived from the Painlevé test for integrability [23]; we note, however, that these exact solutions are unstable to perturbations.

Numerically it was found [17, 24] that stationary pulses are stable over a fairly large range of the control parameter. Afanasjev et al. [25] reported new forms of localized solutions including moving pulses. Moreover, these authors studied the interaction of two pulse solutions in the single cubic-quintic CGL equation. Periodic, quasi-periodic, and chaotic localized solutions have been found by Deissler and Brand [26].

Stationary localized solutions different from pulses have also been found in the cubic-quintic CGL equation. Sakaguchi [27] reported two kinds of hole solutions and showed that their interaction leads to only one type of hole. More recently, Descalzi and Brand showed numerically that the one-dimensional cubic-quintic CGL equation admits five types of stable holes: stationary 2π holes (see figure 1(a)), stationary π holes (see figure 1(b)), moving π holes (moving to the left or to the right) (see figure 1(c)), breathing moving holes (moving to the left or to the right), and breathing non-moving holes [28, 29]. π holes undergo a phase jump by π at the defect location where the modulus $|A|$ vanishes, while the modulus of 2π holes does not touch zero at any point.

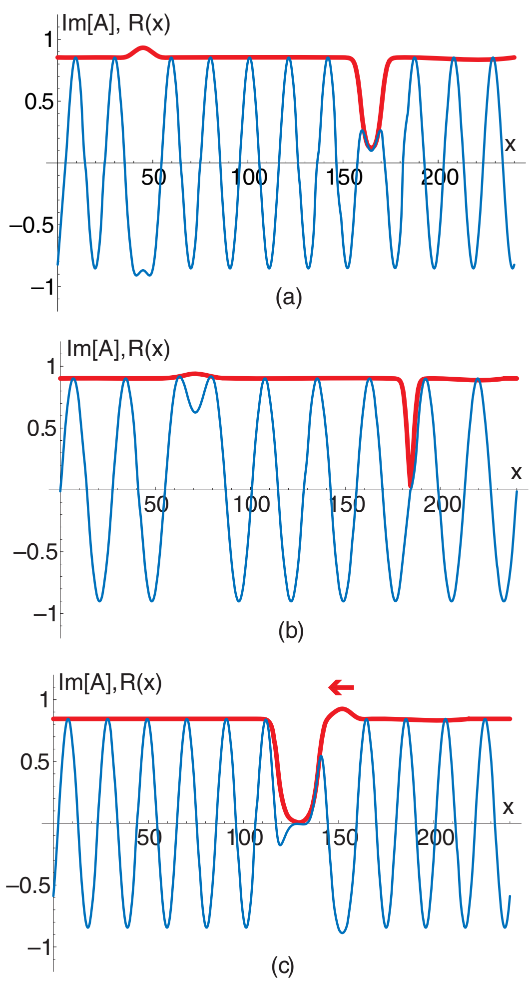

Fig. 1. (a) Stationary 2π hole for $\mu = -0.105$. (b) Stationary π hole for $\mu = -0.100$. (c) Left-moving π hole for $\mu = -0.11345$. Values of the parameters are $\beta_r = -\gamma_r = D_r = 1$, $\beta_i = 0.2$, $\gamma_i = 0.15$, $D_i = -0.1$. The thin continuous line represents $\text{Im}A(x)$ and the thick line stands for the modulus $R(x)$. These figures have been adopted from [29].

3 Coupled cubic-quintic complex Ginzburg–Landau equations

The one-dimensional two coupled cubic-quintic CGL equations for counter-propagating waves, prototype envelope equations near the onset of a weakly inverted (subcritical) instability against traveling waves, can be written as

$$\partial_t A - v\partial_x A = \mu A + (\beta_r + i\beta_i)|A|^2 A + (\gamma_r + i\gamma_i)|A|^4 A + (c_r + ic_i)|B|^2 A + (D_r + iD_i)\partial_{xx} A, \tag{2}$$

$$\partial_t B + v\partial_x B = \mu B + (\beta_r + i\beta_i)|B|^2 B + (\gamma_r + i\gamma_i)|B|^4 B + (c_r + ic_i)|A|^2 B + (D_r + iD_i)\partial_{xx} B, \tag{3}$$

where $A(x,t)$ and $B(x,t)$ are complex fields. For simplicity we have considered cross-coupling terms up to cubic order. To perform a numerical study of these equations we used periodic boundary conditions and the following set of parameters: $\mu = -0.112$, $\beta_r = -\gamma_r = D_r = 1$, $\beta_i = 0.2$, $\gamma_i = 0.15$, $D_i = -0.1$ and $c_i = 0$. We let only two parameters vary, namely, the group velocity v and c_r. With the above mentioned fixed parameters this set of equations admits stable pulses as solutions, and because of the group velocity, pulses are moving in opposite directions. Then we take counter-propagating pulses as an initial condition and we study what are the results of their collision depending on the values of v and c_r. We used a fourth order Runge-Kutta finite differencing numerical method. Very recently, using typically $N = 1000$ points with $\mathrm{d}x = 0.4$ and time step $\mathrm{d}t = 0.1$, we found a phase diagram of possible outcomes which are summarized in figure 2 [16]. From this figure we can see five types of bound states not found in previous investigations: stationary bound states of π holes, and of 2π holes as well as bound states of π holes showing a zigzag motion in space and time. However our most important result was that collision of pulses can lead to holes via front interaction. We also see from the phase diagram that the limit $c_r \to 0$ and $v \to 0$ is unclear and thus it deserves a more detailed analysis.

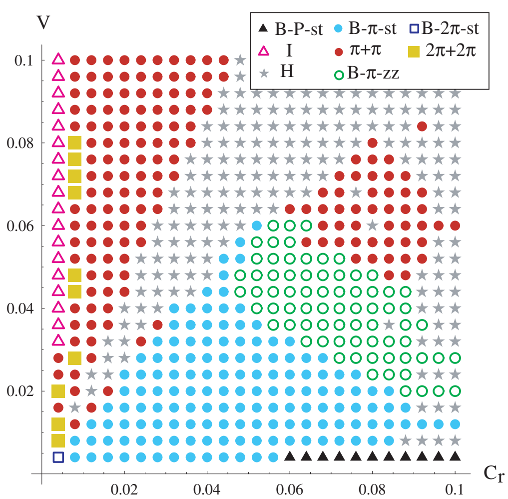

Fig. 2. Phase diagram of possible outcomes resulting from the collision of two stationary pulses while keeping all parameters except for c_r and v fixed. I refers to interpenetration, $B-P-st$ to a stationary bound pair of pulses, $B-2\pi-st$ to a stationary bound pair of 2π holes, $B-\pi-st$ to a stationary bound pair of π holes, $B-\pi-zz$ to a zigzag bound pair of π holes, $\pi+\pi$ to counter-propagating π holes, $2\pi+2\pi$ to counter-propagating 2π holes and H to the spatially homogeneous solution. $\mu = -0.112$, $\beta_r = -\gamma_r = D_r = 1$, $\beta_i = 0.2$, $\gamma_i = 0.15$, $D_i = -0.1$ and $c_i = 0$. This figure has been adopted from [16].

4 Phase diagram and results in the limit of small group velocity and small cubic cross-coupling

Using $N = 2000$ points with $\mathrm{d}x = 0.4$ and time step $\mathrm{d}t = 0.1$ we obtain a blow-up of the open square marked in the bottom left corner of figure 2. The phase diagram with results in the limit of small group velocity and small cubic cross-coupling is shown in figure 3. Further below we will discuss in some detail two examples in this limit. From figures 2 and 3 we see that the possible outcomes after the collision of pulses are: interpenetration, a stationary bound pair of pulses, a zigzag bound pair of pulses, a stationary bound pair of 2π holes, a stationary bound pair of π holes, a zigzag bound pair of π holes, a zigzag bound pair of 2π holes, counter-propagating π holes, counter-propagating 2π holes and the spatially homogeneous solution. We draw the attention of the reader to the rather complex boundary between stationary pairs of π holes and stationary pairs of 2π holes. Such frontiers can be avoided by increasing even further the precision of the phase diagram. We discard the existence of fractal boundaries. In addition, we have observed outcomes we denote by *complex* since collisions of pulses for these parameter values lead to one π hole and one 2π hole, or one π hole and the spatially homogeneous solution. This class of outcomes is not yet fully understood.

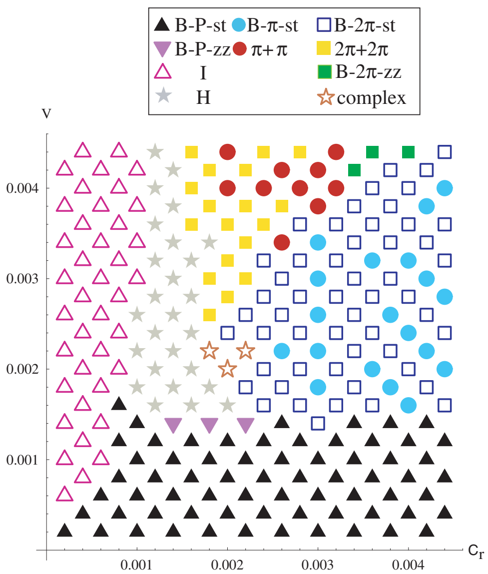

Fig. 3. Phase diagram of possible outcomes resulting from the collision of two stationary pulses for $c_r \to 0$ and $v \to 0$. I refers to interpenetration, $B-P-st$ to a stationary bound pair of pulses, $B-P-zz$ to a zigzag bound pair of pulses, $B-\pi-st$ to a stationary bound pair of π holes, $B-2\pi-st$ to a stationary bound pair of 2π holes, $B-2\pi-zz$ to a zigzag bound pair of 2π holes, $\pi+\pi$ to counter-propagating π holes, $2\pi+2\pi$ to counter-propagating 2π holes and H to the spatially homogeneous solution. $Complex$ refers to an outcome which is only partially understood. $\mu = -0.112$, $\beta_r = -\gamma_r = D_r = 1$, $\beta_i = 0.2$, $\gamma_i = 0.15$, $D_i = -0.1$ and $c_i = 0$.

4.1 Collisions between pulses leading to pulses

Within the above mentioned possible outcomes from the collision between two counter-propagating pulses only three of them are leading to pulses. In figure 3 we see that for small c_r and for any v the outcome becomes interpenetration of pulses, that is, the pulses emerge after the collision with unchanged shapes. For small v and for any c_r the pulses stop moving and form a stationary bound pair of pulses. Both results are known from the literature [11,12]. By varying the parameters c_r and v this stationary compound object can become unstable against a zigzag bound pair of pulses (figure 4). In figure 4(c) the x–t plot shows a complete period ($T = 4615$) for this oscillatory state at $c_r = v = 0.00145$. The underlying mechanism for the transition to the zigzag bound state of two pulses is a Hopf bifurcation. Along the line $c_r = v$ the amplitude and period of the zigzag motion increase with the distance to the bifurcation point showing a supercritical behavior. In this connection we can add that there is no hysteretic behavior of the stationary and zigzag bound states of pulses.

4.2 Collisions between pulses leading to holes

Collisions between pulses leading to holes are very frequent outcomes after the phase diagrams shown in figures 2 and 3. As results we obtained stationary bound states of π holes and 2π

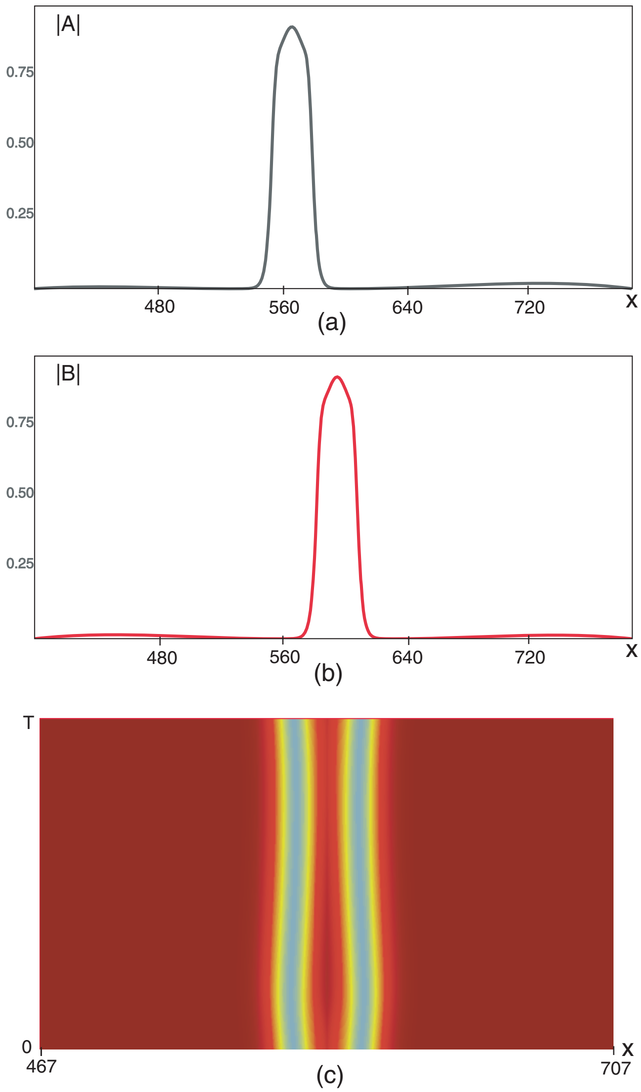

Fig. 4. Zigzag bound pair of pulses for $c_r = v = 0.00145$. (a) and (b) stand for the moduli $|A|$ and $|B|$, respectively, at $t = 0$. (c) $x - t$ plot for $\max(|A|, |B|)$ for one period $T = 4615$. $\mu = -0.112$, $\beta_r = -\gamma_r = D_r = 1$, $\beta_i = 0.2$, $\gamma_i = 0.15$, $D_i = -0.1$ and $c_i = 0$.

holes. These types of behavior occur typically for small v. A Hopf instability for these two states has been observed leading to a zigzag motion in space and time for bound states of π holes and 2π holes. Moreover, collisions of counter-propagating pulses can result in counter-propagating π holes or 2π holes (figure 5). While counter-propagating π holes result rather frequently as a function of the two parameters v and c_r, counter-propagating 2π holes are less frequent as outcome of the collision of two pulses and their appearance is confined to rather smaller values of c_r, $c_r < 0.008$. We also note that the emergence of counter-propagating 2π holes and of counter-propagating π holes as a result of the interaction of pulses gives rise to a rather complex frontier in the phase diagrams.

5 Conclusions

In this paper we have studied the interaction of two counter–propagating pulses for two coupled cubic-quintic complex Ginzburg–Landau equations. Such pulses are well known to arise as locally stable solutions over a large parameter range for a single cubic-quintic complex

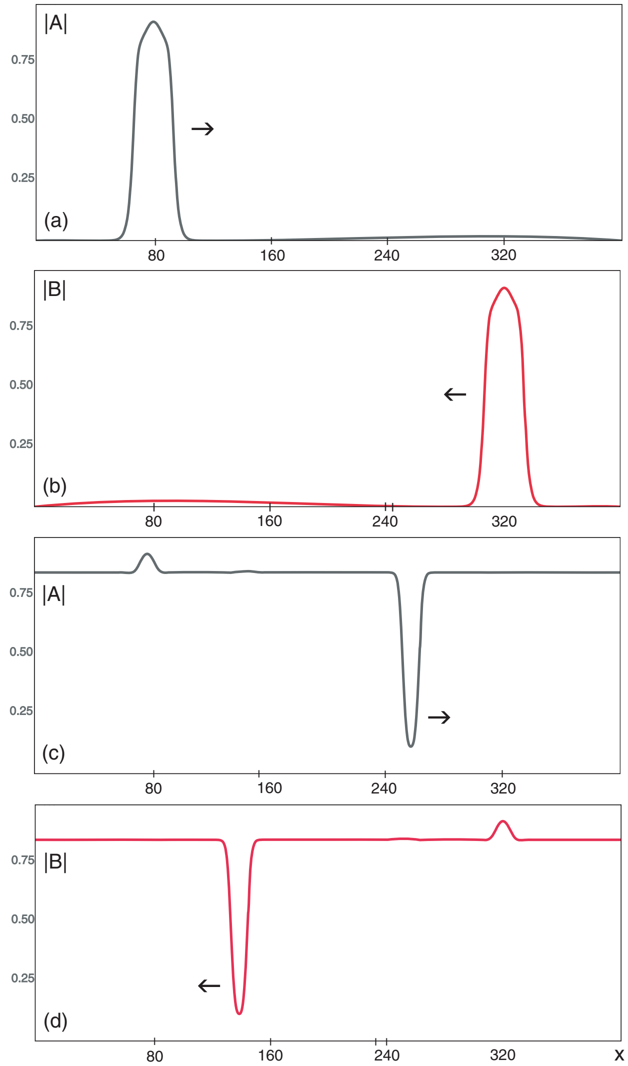

Fig. 5. Collision of counter-propagating pulses leading to counter-propagating 2π holes for $v = 0.048$ and $c_r = 0.008$. (a) and (b) stand for the the moduli $|A|$ and $|B|$ of the initial condition. (c) and (d) stand for the moduli $|A|$ and $|B|$ after the counter-propagating holes have been created via front interaction. $\mu = -0.112$, $\beta_r = -\gamma_r = D_r = 1$, $\beta_i = 0.2$, $\gamma_i = 0.15$, $D_i = -0.1$ and $c_i = 0$.

Ginzburg-Landau equation. We have presented two phase diagrams showing the results of the interaction between two pulses as a function of the pulse velocity and the real part of the cubic cross-coupling. It turns out that the results are particularly rich and complex when the variable parameters are close to zero. As a rather interesting result we find that the zigzag bound pairs of pulses, which arise as a stable solution in this region, have a breathing frequency that is about an order of magnitude smaller than that of bound pairs of π holes and 2π holes. Experimentally one has observed stationary bound pairs of pulses in binary fluid convection [1] and the interpenetration or annihilation of two pulses for surface reactions [7]. For both systems one has also observed experimentally the annihilation of only one pulse. We hope to be able to obtain this outcome also from coupled cubic-quintic CGL equations. It will be most interesting to see which ones of the other predicted phenomena are observed experimentally for systems such as autocatalytic chemical reactions or in the area of dissipative optical solitons. These phenomena include zigzag bound pairs of pulses and holes as well as the generation of counter-propagating π holes.

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